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Normal reactions in a clamped elastic rectangular plate

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Abstract. This paper addresses the classical problem of bending of a clamped thin rectangular elastic plate by a normal uniform load. A brief account of engineering approaches to consider the problem is given. The object of this paper is to provide a full description of the local behaviour of normal reactions and deflection near the corner points of the plate. Among various mathematical and engineering approaches, the method of superposition is effective for solving the problem. Numerical results reveal an analytically predicted local behaviour of normal reactions.

Key words: rectangular elastic plate, biharmonic equation, method of superposition

1. Introduction

This paper considers the distribution of normal reactions along the clamped sides of an elastic rectangular plate under uniform loading. An analytical method is presented and some typical results are discussed in an historical perspective.

The problem of bending of a thin rectangular plate clamped along its edges and subjected to a normal load represents one of these rich mechanical problems that can be considered from both engineering and mathematical points of view. As a rule these approaches are different – a mathematician is usually interested in the problem for its own sake and not for its practical applications, whereas an engineer is interested in the practical problem, using mathematics merely as a tool, see von Kármán [1] for a brilliant discussion. Academician Krylov (or Kriloff in the French spelling of his name), the famous Russian scientist and naval architect, recollected [2] (p. 487):

> In the summer of 1892 I worked in Paris on the project of the Drzewiecki's submarine. Before leaving for Paris, I received from Professor Korkin several of his articles and a letter for Hermite. Upon arrival in Paris, I went to Hermite and was received very warmly. Hermite asked me about Korkin, the Naval Academy, etc. Then I said to Hermite that it would be very important for shipbuilding to obtain a solution of the differential equation with the boundary conditions being that the contour of the plate is fixed. Hermite called his son-in-law Picard and said to him: "Look, Captain Kriloff suggests an excellent topic, which can be used for the *Grand Prix des Mathematiques*. Think about this." Approximately a year later this topic was suggested by the Paris Academy of Sciences.[1]

In the framework of the Kirchhoff two-dimensional linear theory of thin elastic plates, in which all mechanical quantities are expressed in terms of a normal deflection of the middle

[1] This recollection seems not to be completely correct. In 1894 Picard stated the question in the just founded journal *l'Intermédiare des mathématiciens*; and only in 1904 the Paris Academy of Sciences suggested that topic for the competition of the *Prix Vaillant* for the year 1907. For further details see Meleshko [3, 4].

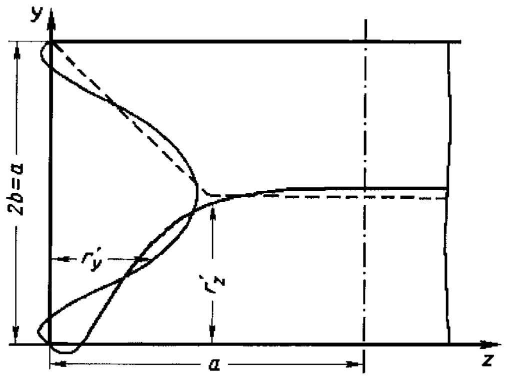

Figure 1. Bubnov's [27] data of distribution of the normal reactions r'_z and r'_y along the contour sides $y = 0$ and $z = 0$ of the clamped plate $0 \leq z \leq 2a$, $0 \leq y \leq 2b$ with aspect ratio $a/b = 2$. Dashed lines correspond to a simply supported plate.

surface only, the problem is reduced to the classical biharmonic problem when function and its normal derivative are prescribed at the boundary.

In mathematics, the biharmonic problem in a rectangular domain has always attracted constant interest. Notably, this problem was the topic of two Presidential Addresses by Love [5] and Dixon [6] delivered before the London Mathematical Society. The main mathematical questions were the solvability of certain functional equations in the complex plane, convergence of series of the nonorthogonal systems of complex eigenfunctions, and the uniqueness of the solution for specific domains with corner points under various boundary conditions. Mathematical results concerning in particular asymptotic approaches, are scolarly and comprehensively described by Ciarlet [7].

In mechanics the problem of bending of a clamped rectangular plate was the main issue of the general opening lecture by Biezeno [8] at the First International Congress for Applied Mechanics (April 23, 1924, Delft, the Netherlands) and of the talk by Timoshenko [9] at the Fifth International Congress for Applied Mechanics (September 13, 1938, Cambridge, USA).

There exists an abundunt literature on the solution of this problem for the clamped rectangular plate. The early textbooks by Grashof [10], Lorenz [11], Timoshenko [12], Föppl and Föppl [13], Prescott [14] considered the problem in details for engineering education. Apart from numerous papers (see Love [15], Nádai [16], Timoshenko [17], Girkmann [18] for a detailed review of early references), already in the beginning of twentieth century the problem was addressed in remarkable dissertations by Koialovich [19], Bubnov [20] (presented early as a talk [21] before the Institution of Naval Architects and published partly in a paper [22]), Ritz [23], and Hencky [24]. Papers by Timoshenko's students appearing in the thirties and forties of the last century (see [17, 25] for references) contain a lot of numerical data of maximum values of deflection and bending moments depending upon the aspect ratio and the type of loading. The textbooks for naval architects by Pietzker [26], Bubnov [27], Shimanski [28], Papkovich [29] present detailed tables with maximum values of normal pressure and total pressures on clamped sides of a uniformly loaded plate.

However, the distributions of normal reactions along the clamped sides and, in particular, the question about their values at the corner points has been addressed more scarcely. Bubnov [27] mentioned some peculiar oscillatory behaviour (Figure 1) of these reactions near the end points of the sides. He related these results to the specific Fourier series employed, 'whose

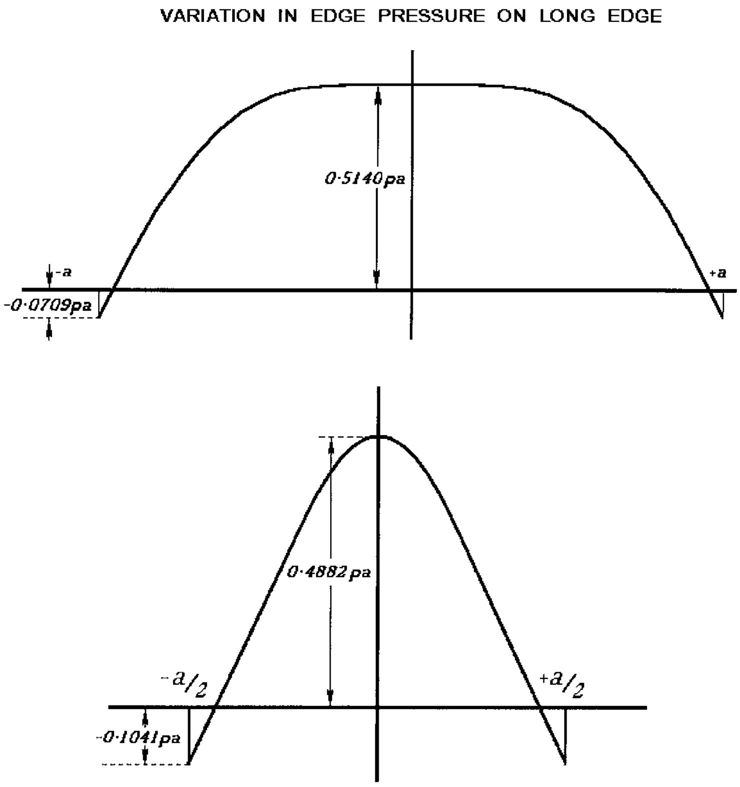

Figure 2. Distribution of the normal reactions along the contour of the clamped plate with aspect ratio $a/b = 2$ according to Inglis's [30] analytical solutions.

convergence leaves much to be desired for practical calculation' ([21], p. 21). On the other hand, Hencky [24] and Inglis [30], using only a few terms in another Fourier series, displayed them (Figure 2) as rather smooth curves. Similar curves were obtained by Galerkin [31] by means of an approximate approach. Woinowsky-Krieger [36] considered an analytical solution for a quarter wedge plate bending by a concentrated force and presented numerical data showing an oscillatory behaviour of normal reactions and shear forces along clamped sides.

There is another interesting aspect of the problem of a clamped rectangular plate which deserves special attention from both a mathematical and an engineering point of view. It concerns the nature of the deflection around a plate corner having two clamped edges. Ritz [23] made the remark that it may not be possible to develop a solution of the governing biharmonic equation into a Taylor series at the corner point. Rayleigh [32] argued that all partial derivatives of the plate deflection w must vanish at the corner point. He concluded that the deflection at a distance r from the corner diminishes more rapidly than any power of r. Dean and Montagnon [33] pointed out that the possibility that w may vary as a fractional power of r appeared to have been overlooked, and in such a case partial derivatives beyond a certain order will be infinite at $r = 0$. These authors (in the context of a mathematically equivalent problem of the Stokes flow) and later Williams [34] discovered that in an *infinite* wedge plate clamped along the sides $\theta = 0$ and $\theta = \alpha$, the non-zero biharmonic function can be written as $w = r^{\lambda+1} f_\lambda(\theta)$, with the values of λ satisfying the equation $\sin^2 \lambda\alpha = \lambda^2 \sin^2 \alpha$.[2] For values of α less than 146·3° all the values of λ are complex with $\Re\lambda > 1{\cdot}83$. With a view of engineering applications, it would be of considerable interest to study such a behaviour of deflection in a *finite* plate. This is a significant and previously unresolved question which the present work takes care of.

The main goal of this paper is to explain the peculiarities of the deflection and the normal reactions near the corner points of the clamped rectangular plate. We apply the analytical solution of the problem by the method of superposition; see [3, 4, 37] for a detailed review of early studies. The final solution involves solving an infinite system of linear algebraic equations providing the relations between an applied loading and the coefficients of two ordinary Fourier series on the complete systems of trigonometric functions in x and y coordinates. It appears that the suggestions resulting from an analysis of the infinite system can be applied for an analytical representation of local plate deflection near the corners. They may also result in a considerable improvement of the algorithm for the calculation of the distribution of normal reactions. We employ the technique based upon Mellin integral transforms, which was developed in [38, 39, 40, 41] for the deduction some general summation formulae. We obtain the results from the solution itself without imposing any preliminary hypothesis about the asymptotic behaviour of the deflection (which can be readily obtained from the well-known solution for an infinite wedge plate).

The paper is organized as follows: the formulation of the problem is outlined in Section 2. A brief summary of the main engineering methods for the solution of the problem is presented in Section 3. The analytical method of superposition is described in Section 4, along with theoretical considerations about the local behaviour of the plate's deflection near the corners and normal reactions at the corner points. Next, Section 5 describes some numerical results concerning the distribution of normal reactions along the clamped sides. Finally, some conclusions are given in Section 6.

2. Statement of the problem

In the classical linear theory of thin plates the differential equation describing the deflection $w(x, y)$ of the middle surface of an elastic isotropic flat rectangular plate of uniform thickness h is

$$D\,\Delta\,\Delta\,w = p. \tag{1}$$

Here Δ denotes the two-dimensional Laplace operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$, the constant $D = Eh^3/12(1-\nu^2)$ is called the flexural rigidity of the plate (E is the Young modulus and ν is the Poisson ratio), $p\,(x, y)$ is the load per unit area of the plate, the coordinates x and y being taken in the plane of the middle surface of the plate before bending.

Two boundary conditions imposed on the function w and its first normal derivatives must also be satisfied. In various engineering structures (bulkheads of a ship, for example) the edges of the plate are firmly clamped, or attached to angle irons which allow no side motions. The deflection w must vanish at the edge; and, in addition, the tangent plane at every point of the edge must remain fixed when the plate is bent.

[2]Remarkably, that the same equation was obtained in the beginning of XXth century by Wieghardt [35] in a benchmark study on elastic stress distribution in a re-entrant corner of an infinite wedge which remains, however, almost unnoticable. Wieghardt studied only the real roots $n = \lambda + 1$ of this equation with $1 \le n \le 2$ for several values of α within the interval $\pi \le \alpha \le 2\pi$.

Thus, if the rectangular plate $|x| \leq a$, $|y| \leq b$ is clamped at all its edges, the boundary conditions are:

$$\begin{aligned} w &= 0, \qquad \frac{\partial w}{\partial x} = 0 \qquad \text{at } x = \pm a, \, |y| \leq b, \\ w &= 0, \qquad \frac{\partial w}{\partial y} = 0 \qquad \text{at } y = \pm b, \, |x| \leq a. \end{aligned} \tag{2}$$

In what follows, for the sake of simplicity, we restrict our consideration of the problem (1)–(2) to the most typical case of uniform loading of intensity p_0 with the deflection w even on both coordinates x and y.

The deflection w defines normal reactions V_x and V_y (sometimes called the amended shearing forces; the quantities opposite in sign are normal pressures on the contour) from the clamped contour's sides $x = a$ and $y = b$, respectively, as follows:

$$\begin{aligned} V_x(y) &= \left[D \frac{\partial \, \Delta w}{\partial x} + D(1-\nu) \, \frac{\partial^3 w}{\partial x \, \partial y^2} \right]_{x=a} \\ V_y(x) &= \left[D \frac{\partial \, \Delta w}{\partial y} + D(1-\nu) \, \frac{\partial^3 w}{\partial x^2 \, \partial y} \right]_{y=b} . \end{aligned} \tag{3}$$

The first terms in the square brackets express the normal shearing forces, and the second ones are the additional shearing forces (according to Kirchhoff and Thomson and Tait) due to the twisting moments in the plate.

The total normal reactions R_x and R_y along the edges $x = a$ and $y = b$ are defined as

$$R_x = \int_{-b}^{b} V_x \,(y) \, \mathrm{d}y, \qquad R_y = \int_{-a}^{a} V_y \,(x) \, \mathrm{d}x. \tag{4}$$

The condition of the static equilibrium of the plate

$$2R_x + 2R_y = 4 \, p_0 \, a \, b \,, \tag{5}$$

must be satisfied.

3. A brief summary of engineering approaches

Several approaches exist to solve approximately the biharmonic problem (1)–(2) governing the plate's deflection w in a convenient manner for direct use in engineering applications. In all of them the boundary conditions (2) are satisfied identically, while Equation (1) is satisfied only to within some approximation. Straightforward calculations according to Equations (3) and (4) with expressions for w provide the normal reactions V_x, V_y along the clamped sides and the total normal reactions R_x, R_y.

3.1. GRASHOF FORMULA

Grashof [10] suggested the approximate expression

$$w = \frac{p_0}{2Eh^3 \, (a^4 + b^4)} \, (a^2 - x^2)^2 \, (b^2 - y^2)^2 \tag{6}$$

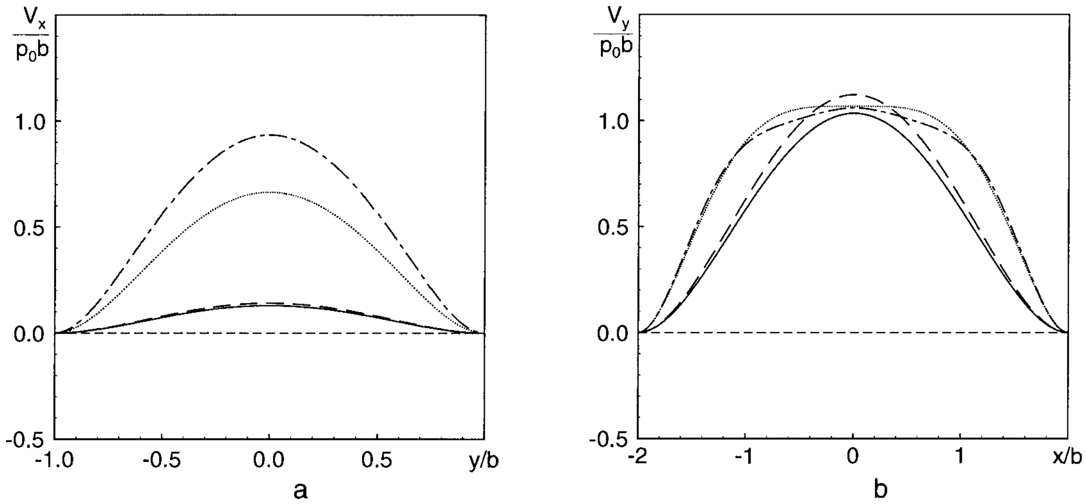

Figure 3. Distribution of the normal reactions along the contour of the clamped plate with aspect ratio $a/b = 2$ according to engineering approaches: (a) normal reaction $V_x(y)$ along the short side $x = a$; (b) normal reaction $V_y(x)$ along the long side $y = b$. Solid line – Grashof [10]; dashed line – energy method, Föppl and Föppl [13] and Leibenzon [47]; short dashed line – energy method, Timoshenko [45] and Lorenz [11, 46]; dot dashed line – Ritz method [23, 44]; dot line – Galerkin method [31].

Table 1. Normal reactions at the clamped sides of the rectangular plate, $a = 2b$, according to engineering approaches.

Source	$\frac{V_x(0)}{p_0 b}$	$\frac{V_x(b)}{p_0 b}$	$\frac{V_y(0)}{p_0 b}$	$\frac{V_y(a)}{p_0 b}$	$\frac{R_x}{p_0 ab}$	$\frac{R_y}{p_0 ab}$	$\frac{R_x + R_y}{p_0 ab}$
Grashof formula [10]	0·129	0	1·034	0	0·069	1·103	1·172
Ritz method [23, 44]	0·935	0	1·059	0	0·518	1·444	1·962
Energy method [47]	0·136	0	1·089	0	0·073	1·162	1·234
Galerkin method [31]	0·665	0	1·067	0	0·462	1·284	1·744
Biezeno method [48]	0·864	0	0·997	0	0·502	1·401	1·903

for the plate occupying the domain $|x| \leq a,\ |y| \leq b$. He considered the plate to be divided into elementary clamped beams parallel to both axes; at any given point the intersecting beams must deflect by the same amount. Due to the assumption of the clamped beam analogy the expression w is independent of ν. (Love [15, Section 314]) gives, however, a formula for w which differs from (6) by a factor $1 - \nu^2$ in the numerator.)

The solid lines in Figure 3 and the first line in Table 1 provide the distribution of reactions and values at some typical points, respectively. Note that Equation (5) is violated essentially.

3.2. Ritz variational method

In his dissertation (published as a benchmark paper [23]) Ritz considered the problem of bending of a clamped rectangular plate as an example of application of his (now famous) variational approach. He chose for the deflection of a rectangular plate $0 \leq x \leq a,\ 0 \leq y \leq b$ the expression

$$w_{MN} = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn}\, \xi_m(x)\, \eta_n(y). \tag{7}$$

Here $\xi_m(x)$ and $\eta_n(y)$ are the eigenfunctions of transverse vibration of the elastic beams $0 \le x \le a$ and $0 \le y \le b$, respectively, satisfying differential equations

$$\frac{\mathrm{d}^4\xi_m}{\mathrm{d}x^4} = \frac{\kappa_m^4}{a^4}\,\xi_m, \qquad \frac{\mathrm{d}^4\eta_n}{\mathrm{d}y^4} = \frac{\kappa_n^4}{b^4}\,\eta_m, \tag{8}$$

with zero boundary conditions on functions and their first derivatives at the ends of intervals. The positive values of κ_m and κ_n are the roots of the equation $\cos\kappa\ \cosh\kappa = 1$.

Minimizing procedure for the functional

$$J_{MN} = \int_0^b \int_0^a [\tfrac{1}{2}(\Delta w_{MN})^2 - f w_{MN}]\, \mathrm{d}x\, \mathrm{d}y \tag{9}$$

(with $f = p/D$) leads after some obvious transformations to the system of linear algebraic equation for the coefficients a_{mn}

$$\frac{\partial J_{MN}}{\partial a_{mn}} = \int_0^b \int_0^a \left[(\Delta w_{MN})\, \Delta(\xi_m\, \eta_n) - f\, \xi_m\, \eta_n\right]\, \mathrm{d}x\, \mathrm{d}y = 0, \quad (m = 1, 2, \ldots, M, \quad n = 1, 2, \ldots, N). \tag{10}$$

By twice integrating by parts and taking into account the zero boundary conditions for functions ξ_m, η_n and their first derivatives, Ritz arrived to the following system of linear algebraic equation for the coefficients a_{mn}

$$\int_0^b \int_0^a (\Delta\, \Delta\, w_{MN} - f\,)\, \xi_m\, \eta_n\, \mathrm{d}x\, \mathrm{d}y = 0, \quad (m = 1, 2, \ldots, M, \quad n = 1, 2, \ldots, N). \tag{11}$$

It is worth mentioning that the system of Equations (11) (number (41) on page 38 of the original paper [23]) represents nothing else but the essence of the so-called Galerkin [31] method. This was already noted by Biezeno [8, 42] and later mentioned in [25, Section 81] (see also [3, 4] for further historical details).

Ritz performed extensive calculations for a square plate under uniform load p_0 and presented three approximations for the deflection.

Poincaré in his preface to the posthumous Ritz's collected papers ([43], p. XVI) highly esteemed this approach and named it 'une méthode d'ingénieur'. He stressed that for concrete calculations it is much preferable to the rigorious method based upon Fredholm's integral equations.

Ritz's calculations were repeated and extended to several other values of a/b ratio in the diploma work [44] of Pistriakoff, a student of Timoshenko at the Kiev Polytechnic Institute. Using his expression we calculated the distribution of the reactions and their total values for a clamped plate with $a/b = 2$ ratio. These results are presented in the second line in the Table 1 and as the dot-dashed lines in Figure 3.

3.3. Energy methods

Timoshenko [45] and Lorenz [11, 46] used the energy method that consists in equalizing the potential energy of bending to the work done by the external loading under prescribed plate deflection compatible with boundary conditions (2). For a clamped plate $0 \le x \le a$, $0 \le y \le b$ chosing the trigonometric functions as trial ones they obtained the deflection as following:

$$w = \frac{p_0 \, a^4 \, b^4}{4\pi^4 \, D \left(3a^4 + 3b^4 + 2a^2 \, b^2\right)} \left(1 - \cos \frac{2\pi x}{a}\right) \left(1 - \cos \frac{2\pi y}{b}\right). \tag{12}$$

This solution (short-dashed lines in Figure 3) leads to the paradoxical result of zero reactions along the sides.

By the same approach, but with different approximation functions, father and son Föppl [13] and Leibenzon [47] obtained for the deflection w in a plate $|x| \le a$, $|y| \le b$ the expression:

$$w = \frac{7 p_0}{128 \, D \left(a^4 + b^4 + \frac{4}{7} a^2 b^2\right)} \, (x^2 - a^2)^2 \, (y^2 - b^2)^2. \tag{13}$$

Dashed lines in Figure 3 and the third line in Table 1 show the distribution of reactions according to this version of the energy approach.

3.4. Galerkin method

In his widely cited paper [31] Galerkin considered bending by uniform load of a clamped plate $|x| \le a/2$, $|y| \le b/2$ and chose the deflection in the form

$$w = \sum_{k=2}^{3} \sum_{n=2}^{3} A_{kn} \, (a^2 - 4x^2)^k \, (b^2 - 4y^2)^n \, . \tag{14}$$

(It is worth noting, that Ritz [23] has also mentioned the expression (14) as a possible choice of trial functions, but he did not perform any further calculations.)

Galerkin substituted the expression (14) in Equation (1), multiplied both sides of it subsequently by $(a^2 - 4x^2)^k \, (b^2 - 4y^2)^n \, \mathrm{d}x \, \mathrm{d}y$, with $k = 2, 3$ and $n = 2, 3$ and integrated over the plate area. In that way he obtained a linear system of four algebraic equations for defining the values of A_{kn}. Galerkin solved that system for three values of the ratio $a/b = 1, 1{\cdot}5, 2$. The dotted lines in Figure 3 and the fourth line in Table 1 represent our calculations of the distribution of normal reactions according to this approximate approach.

3.5. Biezeno and Koch method

Biezeno and Koch [48], see also [8, 49], suggested a modification of the Galerkin procedure in the following way. The took the representation of the deflection in the clamped plate $|x| \le a$, $|y| \le b$ (we have slightly changed the notations of the original publication) as

$$w = \left(\frac{x^2}{a^2} - 1\right)^2 \left(\frac{y^2}{b^2} - 1\right)^2 \sum_{k=1}^{2} \sum_{n=1}^{2} f_{kn} \left(\frac{x^2}{a^2}\right)^{n-1} \left(\frac{y^2}{b^2}\right)^{k-1} , \tag{15}$$

which satisfies the boundary conditions (2) and substituted it in the governing Equation (1). In such a way some fictive load $\bar{p}$ instead of p has appeared in the right-hand side of Equation (1). Then, the coefficients f_{kn} are determined by the conditions that the integrals $\int\int p \, \mathrm{d}x \, \mathrm{d}y$ and $\int\int \bar{p} \, \mathrm{d}x \, \mathrm{d}y$, taken over a well-chosen region of the plate surface, are equal. In the particular case of a uniform load p_0, a quadrant of the plate limited by the two axes $x = 0$ and $y = 0$ and by the lines $x = a$ and $y = b$, was subdivided into four parts by the lines $x = \frac{1}{2}a$ and $y = \frac{1}{2}b$. After a laborious but straightforward calculation of the integrals, a system of four equations which determines the coefficients f_{kn} was obtained. In the original paper [48] only the ratio $a/b = 1{\cdot}5$ was considered in full detail. We performed the calculations for the ratio $a/b = 2$ and the results for the reactions are presented in the fifth line of Table 1.

4. Method of superposition and analytical results

The analytical solution of the boundary-value problem (1)–(2) can be constructed by means of the method of superposition in two different ways, depending on which boundary conditions for the w function or its normal derivative are to be satisfied identically. In what follows we give only one representation of the solution, first introduced by Hencky [24] and later used by Timoshenko [9, 12], Sezawa [50], March [51], Inglis [30], Love [15] and many others. The second representation of the solution introduced by Koialovich [19] and Bubnov [22, 27] was already discussed in detail in our previous papers [3, 4, 37].

The representation

$$\begin{aligned} w = \frac{p_0}{8D} \Bigg\{ & \left(a^2 - x^2\right)\left(b^2 - y^2\right) \\ & -b \sum_{m=1}^{\infty} (-1)^m \frac{X_m}{\alpha_m} \left(b \tanh \alpha_m b \frac{\cosh \alpha_m y}{\cosh \alpha_m b} - y \frac{\sinh \alpha_m y}{\cosh \alpha_m b} \right) \cos \alpha_m x \\ & +a \sum_{l=1}^{\infty} (-1)^l \frac{Y_l}{\beta_l} \left(a \tanh \beta_l a \frac{\cosh \beta_l x}{\cosh \beta_l a} - x \frac{\sinh \beta_l x}{\cosh \beta_l a} \right) \cos \beta_l y \Bigg\}, \end{aligned} \tag{16}$$

with the notations

$$\alpha_m = \frac{(2m-1)\,\pi}{2a}, \qquad \beta_l = \frac{(2l-1)\,\pi}{2b},$$

identically satisfies with arbitrary coefficients X_m, Y_l the inhomogeneous biharmonic equation (1) and the boundary conditions (2) for the function itself. The conditions (2) for the normal derivatives by means of the Fourier expansions

$$h^2 - z^2 = \sum_{p=1}^{\infty} (-1)^{p-1} \frac{4}{h\,\delta_p^3} \cos \delta_p z, \qquad \delta_p = \frac{(2p-1)\pi}{2h},$$

$$h \tanh ch \frac{\cosh cz}{\cosh ch} - z \frac{\sinh cz}{\cosh ch} = \sum_{p=1}^{\infty} (-1)^{p-1} \frac{4c\delta_p}{h\,(\delta_p^2 + c^2)^2} \cos \delta_p z\,,$$

provides the following infinite system of linear algebraic equations for defining the coefficients X_m, Y_l:

$$\left.\begin{aligned} X_m\, b\, \Delta(\alpha_m b) &= \sum_{l=1}^{\infty} Y_l \frac{4\alpha_m^2 \beta_l}{\left(\beta_l^2+\alpha_m^2\right)^2} - \frac{8b}{a\alpha_m^2}\,, \quad m=1,2,\ldots \\ Y_l\, a\, \Delta(\beta_l a) &= \sum_{m=1}^{\infty} X_m \frac{4\beta_l^2 \alpha_m}{\left(\alpha_m^2+\beta_l^2\right)^2} + \frac{8a}{b\beta_l^2}\,, \quad l=1,2,\ldots \end{aligned}\right\}, \tag{17}$$

where

$$\Delta(\xi) = \tanh\xi + \frac{\xi}{\cosh^2\xi}\,.$$

The infinite system of Equations (17) is the *fully regular* one, that is, the sum of off-diagonal elements of *each* row of right-hand side infinite matrix is less than the corresponding diagonal element by some positive value. Therefore, according to general theory (see Kantorovich and Krylov [52] for details) it has a unique bounded (the so-called principal) solution. A traditional method of numerical solution of the regular infinite system consists in the simple reduction approach by putting

$$X_m = 0, \qquad m > M, \qquad\qquad Y_l = 0, \qquad l > L, \tag{18}$$

leaving in Equations (17) the first M and L equations, and then solving the finite system by any technique (in particular, the algorithm of successive approximations is rapidly convergent).

The asymptotic behaviour of the unknowns is established in [37]

$$X_m = \mathfrak{Re}\left(E_\gamma\, \alpha_m^{-\gamma_1}\right) + x_m\,, \qquad Y_l = -\mathfrak{Re}\left(E_\gamma\, \beta_l^{-\gamma_1}\right) + y_l\,, \tag{19}$$

with

$$x_m = o\left(m^{-3}\right), \quad m\to\infty \qquad y_l = o\left(l^{-3}\right), \qquad l\to\infty\,. \tag{20}$$

Here E_γ is a complex constant, $\gamma_1 = 2{\cdot}73959 + \mathrm{i}\,1{\cdot}11902$ is the root of equation

$$\sin\tfrac{1}{2}\gamma\pi + \gamma = 0 \tag{21}$$

with the lowest positive real part.

This asymptotic behaviour provides the solid basis for solving the infinite system of Equations (17) by the simple reduction method and, consequently, for all numerical results referred to in [17].

4.1. Normal reactions

For the representation (16) the normal reactions at the clamped sides $x = a$ and $y = b$, respectively, can be written as

$$\begin{aligned} V_x(y) = \frac{p_0}{4}\Bigg[2a &+ b\sum_{m=1}^{\infty} X_m\,\alpha_m \frac{\cosh\alpha_m y}{\cosh\alpha_m b} \\ &- a\sum_{l=1}^{\infty}(-1)^l\, Y_l\,\beta_l\,\tanh\beta_l a\,\cos\beta_l y\Bigg], \\ V_y(x) = \frac{p_0}{4}\Bigg[2b &+ b\sum_{m=1}^{\infty}(-1)^m X_m\,\alpha_m\,\tanh\alpha_m b\,\cos\alpha_m x \\ &- a\sum_{l=1}^{\infty} Y_l\,\beta_l \frac{\cosh\beta_l x}{\cosh\beta_l a}\Bigg]. \end{aligned} \tag{22}$$

Note, that the additional shearing forces in (3) contribute nothing to the expressions (22).

The total reactions along these sides are

$$R_x = \frac{p_0}{2}\Big(2ab + b\sum_{m=1}^{\infty} X_m \tanh\alpha_m b + a\sum_{l=1}^{\infty} Y_l \tanh\beta_l a\Big),$$

$$R_y = \frac{p_0}{2}\Big(2ab - b\sum_{m=1}^{\infty} X_m \tanh\alpha_m b - a\sum_{l=1}^{\infty} Y_l \tanh\beta_l a\Big). \tag{23}$$

In spite of the complicated appearance of these total reactions Equation (5) is identically satisfied.

Because of the asymptotic behaviour (19) of the Fourier coefficients X_m and Y_l, expression (16) represents the deflection w which is a thrice differentiable function in the whole domain $|x| \le a$, $|y| \le b$ *including* boundary with the corner points. Therefore, by taking limits along the respective clamped sides we conclude that *all* partial derivatives of w up to third order at corner points are equal to zero. Apparently, Bubnov had these reasons in mind when he noticed [27, Section 20] that 'some reasons which we are not going to present incline to think that at these four [corner] points the pressure will be zero.' Rayleigh [32] went further using similar arguments and arrived at the erroneous conclusion that the deflection w at a distance r from the corner should diminish more rapidly than any power of r.

Thus, according to definitions (3) we obtain

$$V_x(b) = 0, \qquad V_y(a) = 0. \tag{24}$$

On the other hand, at the corner point (a, b) we have from expressions (22) for the normal reactions

$$V_x(b) = \frac{p_0 b}{16}\left(\sum_{m=1}^{\infty} 4\, X_m\, \alpha_m + \frac{8a}{b}\right), \quad V_y(a) = -\frac{p_0 a}{16}\left(\sum_{l=1}^{\infty} 4\, Y_l\, \beta_l - \frac{8b}{a}\right). \tag{25}$$

By multiplying the m-th equation in the first line of Equations (17) by α_m^2, the l-th equation in the second line of Equations (17) by β_l^2, and considering the limiting equations for $m \to \infty$ and $l \to \infty$, we conclude that their right-hand sides are identical to the expressions in brackets of (25). The left-hand sides of these 'infinite' equations tend to zero because of the asymptotic law (19).

The infinite system of Equations (17) can obviously be written in terms of the unknowns x_m, y_l and E_γ as:

$$\left.\begin{aligned}
x_m\, b\, \Delta(\alpha_m b) = {} & \sum_{l=1}^{\infty} y_l \frac{4\alpha_m^2 \beta_l}{(\beta_l^2 + \alpha_m^2)^2} - \frac{8b}{a\alpha_m^2} \\
& - \mathfrak{Re}\left\{E_\gamma \left[\frac{b\,\Delta(\alpha_m b)}{\alpha_m^{\gamma_1}} + \sum_{l=1}^{\infty} \frac{4\alpha_m^2}{\beta_l^{\gamma_1 - 1}(\beta_l^2 + \alpha_m^2)^2}\right]\right\}, \ m = 1, 2, \ldots \\
y_l\, a\, \Delta(\beta_l a) = {} & \sum_{m=1}^{\infty} x_m \frac{4\beta_l^2 \alpha_m}{(\alpha_m^2 + \beta_l^2)^2} + \frac{8a}{b\beta_l^2} \\
& + \mathfrak{Re}\left\{E_\gamma \left[\frac{a\,\Delta(\beta_l a)}{\beta_l^{\gamma_1}} + \sum_{m=1}^{\infty} \frac{4\beta_l^2}{\alpha_m^{\gamma_1 - 1}(\alpha_m^2 + \beta_l^2)^2}\right]\right\}, \ l = 1, 2, \ldots
\end{aligned}\right\} \tag{26}$$

Two additional equations for defining E_γ can be obtained from Equations (24)–(25) as

$$\left.\begin{aligned} &\sum_{m=1}^{\infty} x_m\,\alpha_m + \mathfrak{Re}\left(E_\gamma \sum_{m=1}^{\infty}\frac{1}{\alpha_m^{\gamma_1-1}}\right) + 2ab = 0,\\ &\sum_{l=1}^{\infty} y_l\,\beta_l - \mathfrak{Re}\left(E_\gamma \sum_{l=1}^{\infty}\frac{1}{\beta_l^{\gamma_1-1}}\right) - 2ba = 0. \end{aligned}\right\} . \tag{27}$$

To study the local behaviour of $V_x(y)$ and $V_y(x)$ near the corner point let us proceed as follows. By means of Equations (19) and the identities

$$\tanh\alpha b = 1 + 2\sum_{k=1}^{\infty}(-1)^k\,\mathrm{e}^{-2k\alpha b}\,, \tag{28}$$

$$\frac{\cosh\beta x}{\cosh\beta a} = \mathrm{e}^{-\beta(a-x)} + \sum_{k=1}^{\infty}(-1)^k\left[\mathrm{e}^{-\beta(2ka+a-x)} - \mathrm{e}^{-\beta(2ka-a+x)}\right]\,, \tag{29}$$

we can present V_y as

$$V_y = \frac{p_0}{4}\left[2b + V_y^{(0)} + V_y^{(E)} + V_y^{(\gamma)}\right]\,, \tag{30}$$

where

$$V_y^{(0)} = b\sum_{m=1}^{\infty}(-1)^m x_m\alpha_m \tanh\alpha_m b\cos\alpha_m x - a\sum_{l=1}^{\infty} y_l\,\beta_l\,\frac{\cosh\beta_l x}{\cosh\beta_l a}, \tag{31}$$

$$V_y^{(E)} = \mathfrak{Re}\left\{E_\gamma\,R(x)\right\}\,,\quad V_y^{(\gamma)} = \mathfrak{Re}\left\{E_\gamma\,\left[aS_\beta(x) + bS_\alpha(x)\right]\right\}\,, \tag{32}$$

with the notations

$$\begin{aligned} R(x) = \sum_{k=1}^{\infty}(-1)^k &\left\{\sum_{m=1}^{\infty}(-1)^m\,2b\alpha_m^{\gamma_1-1}\mathrm{e}^{-2k\alpha_m b}\cos\alpha_m x\right.\\ &\left.+\sum_{l=1}^{\infty} a\beta_l^{\gamma_1-1}\left[\mathrm{e}^{-\beta_l(2ka+a-x)} - \mathrm{e}^{-\beta_l(2ka-a+x)}\right]\right\}\,, \end{aligned} \tag{33}$$

$$S_\alpha(x) = \sum_{m=1}^{\infty}\frac{(-1)^m}{\alpha_m^{\gamma_1-1}}\cos\alpha_m x\,,\quad S_\beta(x) = \sum_{l=1}^{\infty}\frac{\mathrm{e}^{-\beta_l(a-x)}}{\beta_l^{\gamma_1-1}}\,. \tag{34}$$

Due to the asymptotic behaviour (20) of x_m and y_l, the series in the representation $V_y^{(0)}$ can be expanded into a Taylor series on $\xi = a - x$ at the point $x = a$. We obtain

$$V_y^{(0)} = -a\sum_{l=1}^{\infty} y_l\,\beta_l + \mathcal{O}(\xi)\,. \tag{35}$$

Similarly, we can write the Taylor expansion as follows

$$R(x) = R_1\,\xi + \mathcal{O}(\xi^2)\,, \tag{36}$$

with

$$R_1 = a \sum_{l=1}^{\infty} \frac{\tanh \beta_l\, a - 1}{\beta_l^{\gamma_1 - 2}} + b \sum_{m=1}^{\infty} \frac{\tanh \alpha_m\, b - 1}{\alpha_m^{\gamma_1 - 2}} \,,$$

(here we use Equation (28) for summation over k), and, consequently, the leading term in $V^{(E)}$ is

$$V_y^{(E)} = \mathfrak{Re} \left\{ E_\gamma\, R_1 \right\} \xi + \mathcal{O}(\xi^2)\,. \tag{37}$$

This procedure, however, can not be employed to the series expressing the functions $S_\alpha(x)$ and $S_\beta(x)$. To obtain the local expansions of these functions near the point $x = a$ we use the following expansions

$$\sum_{l=1}^{\infty} \frac{e^{-(l-\frac{1}{2})c}}{\left(l - \frac{1}{2}\right)^{\mu}} = \Gamma(1-\mu)c^{\mu-1} + \zeta(\mu, \tfrac{1}{2}) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \zeta(\mu - n, \tfrac{1}{2}) c^n \,,$$

$$\sum_{m=1}^{\infty} \frac{\sin(m - \frac{1}{2})c}{\left(m - \frac{1}{2}\right)^{\mu}} = \Gamma(1-\mu)\, 2^{-\mu} \sin \frac{\pi(1-\mu)}{2}\, c^{\mu-1} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sin \frac{n\pi}{2}\, \zeta(\mu - n, \tfrac{1}{2}) \left(\frac{c}{2}\right)^n \,,$$

with $\mathfrak{Re}\mu > 0,\ \mathfrak{Re}\mu \neq 1, 2, \ldots$ and $c > 0$. Here $\Gamma(z)$ denotes the gamma function and

$$\zeta(z, \tfrac{1}{2}) = \sum_{m=0}^{\infty} (m + \tfrac{1}{2})^{-z} \tag{38}$$

is the generalized Riemann ζ function. These expansions can be readily obtained by the Mellin transforms technique, see [38, 39, 40, 41] for details.

Therefore we have

$$S_\alpha(x) = -\frac{a}{\pi}\, \Gamma(2 - \gamma_1)\, \sin \frac{\pi \gamma_1}{2}\, \xi^{\gamma_1 - 2} + \mathcal{O}(\xi)\,,$$

$$S_\beta(x) = \left(\frac{b}{\pi}\right)^{\gamma_1 - 1} \zeta\left(\gamma_1 - 1, \tfrac{1}{2}\right) + \frac{b}{\pi}\, \Gamma(2 - \gamma_1) \xi^{\gamma_1 - 2} + \mathcal{O}(\xi)\,,$$

and Equation (30) can be written as

$$\begin{aligned} V_y(x) = {} & \frac{p_0}{4} \left[2b - a \sum_{l=1}^{\infty} y_l\, \beta_l + \mathfrak{Re} \left\{ E_\gamma \left(\frac{b}{\pi}\right)^{\gamma_1 - 1} \zeta\left(\gamma_1 - 1, \tfrac{1}{2}\right) \right\} \right] \\ & + \frac{p_0\, ab}{4\pi}\, \mathfrak{Re} \left\{ E_\gamma\, \Gamma(2 - \gamma_1) \left(1 - \sin \tfrac{1}{2}\pi\gamma_1\right) (a - x)^{\gamma_1 - 2} \right\} + \mathcal{O}(a - x)\,. \end{aligned} \tag{39}$$

The first term by means of (38) is the left-hand side of the second Equation (27) and, consequently, it turns out to be zero. Thus, the asymptotic expansion of $V_y(x)$ when $x \to a$ is

$$V_y\,(x) = \frac{p_0\, ab}{4\pi}\, \mathfrak{Re} \left\{ E_\gamma\, \Gamma(2 - \gamma_1)(1 + \gamma_1)\, (a - x)^{\gamma_1 - 2} \right\} + \mathcal{O}(a - x)\,. \tag{40}$$

In a similar way one can establish the leading terms of the asymptotic expansions of V_x for $y \to b$:

$$V_x\,(y) = \frac{p_0\,ab}{4\pi}\,\Re\mathfrak{e}\left\{E_\gamma\,\Gamma(2-\gamma_1)(1+\gamma_1)\,(b-y)^{\gamma_1-2}\right\} + \mathcal{O}(b-y)\,. \tag{41}$$

Thus, the normal reactions along the contour behave in a rather complicated manner: they are zero at the corner points, and they have infinitely many changes of sign near these points because of the complex power γ_1 in Equations (40) and (41).

4.2. Deflection of the plate near a corner point

To investigate the local behaviour of the deflection w near the corner point (a, b) let us represent the deflection in terms of coefficients x_m, y_l and E_γ

$$w = \frac{p_0}{8D}\,\left(w_0 + w_E + w_\gamma\right)\,, \tag{42}$$

where

$$\begin{aligned} w_0 = &\left(a^2 - x^2\right)\left(b^2 - y^2\right) \\ &-b\sum_{m=1}^{\infty}(-1)^m\,\frac{x_m}{\alpha_m}\left(b\tanh\alpha_m\,b\,\frac{\cosh\alpha_m\,y}{\cosh\alpha_m\,b} - y\,\frac{\sinh\alpha_m\,y}{\cosh\alpha_m\,b}\right)\cos\alpha_m\,x \\ &+a\sum_{l=1}^{\infty}(-1)^l\,\frac{y_l}{\beta_l}\left(a\tanh\beta_l\,a\,\frac{\cosh\beta_l\,x}{\cosh\beta_l\,a} - x\,\frac{\sinh\beta_l\,x}{\cosh\beta_l\,a}\right)\cos\beta_l\,y\,, \end{aligned} \tag{43}$$

$$w_E = \Re\mathfrak{e}\left\{E_\gamma\,\left[W_1(x,y) + W_2(x,y)\right]\right\}\,, \tag{44}$$

$$w_\gamma = \Re\mathfrak{e}\left\{E_\gamma\,W_\gamma(x,y)\right\}\,, \tag{45}$$

with the notations

$$\begin{aligned} W_1(x,y) = \;& b\sum_{k=1}^{\infty}(-1)^{k-1}\,\big[(2kb+b-y)\,S_1\,(x, 2kb+b-y) \\ &-(2kb-b+y)\,S_1\,(x, 2kb-b+y)\big]\,, \end{aligned} \tag{46}$$

$$\begin{aligned} W_2(x,y) = \;& a\sum_{k=1}^{\infty}(-1)^{k-1}\big[(2ka+a-x)\,S_2\,(2ka+a-x, y) \\ &-(2ka-a+x)\,S_2\,(2ka-a+x, y)\big]\,, \end{aligned} \tag{47}$$

$$\begin{aligned} W_\gamma(x,y) = \;& b\,(y-b)\sum_{m=1}^{\infty}\frac{(-1)^m}{\alpha_m^{\gamma_1+1}}\,\mathrm{e}^{-\alpha_m\,(b-y)}\,\cos\alpha_m\,x \\ &+a\,(x-a)\sum_{l=1}^{\infty}\frac{(-1)^l}{\beta_l^{\gamma_1+1}}\,\mathrm{e}^{-\beta_l\,(a-x)}\,\cos\beta_l\,y\,, \end{aligned} \tag{48}$$

$$S_1\,(x,\eta) = \sum_{m=1}^{\infty}\frac{(-1)^m}{\alpha_m^{\gamma_1+1}}\,\mathrm{e}^{-\alpha_m\,\eta}\,\cos\alpha_m\,x\,, \tag{49}$$

$$S_2\,(\xi, y) = \sum_{l=1}^{\infty}\frac{(-1)^l}{\beta_l^{\gamma_1+1}}\,\mathrm{e}^{-\beta_l\,\xi}\,\cos\beta_l\,y\,. \tag{50}$$

In these transformations we used the identity

$$h\,\tanh ch\,\frac{\cosh cz}{\cosh ch} - z\,\frac{\sinh cz}{\cosh ch} = (h-z)\,\mathrm{e}^{-c\,(h-z)}$$
$$+\sum_{k=1}^{\infty}(-1)^k\left[(2kh+h-z)\,\mathrm{e}^{-c\,(2kh-h+z)} - (2kh-h+z)\,\mathrm{e}^{-c\,(2kh-h+z)}\right].$$

Next, let us introduce local polar coordinates (ρ,θ) in the vicinity of the corner point (a,b)

$$x = a - \rho\cos\theta, \qquad y = b - \rho\sin\theta,$$

with $\rho \geq 0$, $0 \leq \theta \leq \frac{1}{2}\pi$.

Due to the asymptotic behaviour (20) of the Fourier coefficients the expression w_0 represents a smooth, at least three times differentiable function in the whole rectangular domain including its boundary. In polar coordinates expanding into the Taylor series on ρ the trigonometric and exponential terms in series (43) after some obvious algebra we obtain:

$$\begin{aligned} w_0(\rho,\theta) = &\left[b\sum_{m=1}^{\infty} x_m\,\Delta(\alpha_m\,b) - a\sum_{l=1}^{\infty} y_l\,\Delta(\beta_l\,a) + 4ab\right]\rho^2\sin\theta\cos\theta \\ &-\left[b\sum_{m=1}^{\infty} x_m\,\alpha_m + 2a\right]\rho^3\sin^2\theta\cos\theta \\ &+\left[a\sum_{l=1}^{\infty} y_l\,\beta_l - 2a\right]\rho^3\sin\theta\cos^2\theta + \mathcal{O}\left(\rho^4\right). \end{aligned} \tag{51}$$

The functions W_1 and W_2 also admit the expansions of the series into the Taylor series. Straightforward transformations by using the identity (28) for performing summation over k provide

$$\begin{aligned} W_1(\rho,\theta) &= \left[b\sum_{m=1}^{\infty}\frac{\Delta(\alpha_m\,b)-1}{\alpha_m^{\gamma_1}}\right]\rho^2\sin\theta\cos\theta + \mathcal{O}\left(\rho^4\right), \\ W_2(\rho,\theta) &= \left[a\sum_{l=1}^{\infty}\frac{\Delta(\beta_l\,a)-1}{\beta_l^{\gamma_1}}\right]\rho^2\sin\theta\cos\theta + \mathcal{O}\left(\rho^4\right), \end{aligned} \tag{52}$$

and, consequently

$$\begin{aligned} w_E(\rho,\theta) = &\,\mathfrak{Re}\left\{E_\gamma\left[b\sum_{m=1}^{\infty}\frac{\Delta(\alpha_m\,b)-1}{\alpha_m^{\gamma_1}} + a\sum_{l=1}^{\infty}\frac{\Delta(\beta_l\,a)-1}{\beta_l^{\gamma_1}}\right]\right\} \\ &\cdot\rho^2\sin\theta\cos\theta + \mathcal{O}\left(\rho^4\right). \end{aligned} \tag{53}$$

We can represent $W_\gamma(x,y)$ in Equation (48) in polar coordinate as

$$\begin{aligned} W_\gamma(\rho,\theta) = &\,b\,\rho\sin\theta\sum_{m=1}^{\infty}\frac{\mathrm{e}^{-\alpha_m\,\rho\sin\theta}}{\alpha_m^{\gamma_1+1}}\sin(\alpha_m\,\rho\cos\theta) \\ &+a\,\rho\cos\theta\sum_{l=1}^{\infty}\frac{\mathrm{e}^{-\beta_l\,\rho\cos\theta}}{\beta_l^{\gamma_1+1}}\sin(\beta_l\,\rho\sin\theta) \end{aligned} \tag{54}$$

To analyse the behavior of the function $W_\gamma(\rho,\theta)$ when $\rho \to 0$ we use the Mellin transform technique. For the Mellin transform $U(s,\theta)$ of the function $W_\gamma(\rho,\theta)$

$$U(s,\theta) = \int_0^\infty W_\gamma(\rho,\theta)\rho^{s-1}\,\mathrm{d}\rho \tag{55}$$

by means of the value of integral [53, p. 45]

$$\int_0^\infty \mathrm{e}^{-px}\,\sin qx\, x^{z-1}\,\mathrm{d}x = \Gamma(z)\left(p^2+q^2\right)^{-\frac{1}{2}z}\,\sin[z\arctan(q/p)]$$

with $\mathfrak{Re}\, z > -1,\; p > 0,\; q > 0$, we obtain

$$\begin{aligned} U(s,\theta) = {} & \frac{ab}{\pi}\,\Gamma(s+1)\,\zeta(s+\gamma_1+2,\tfrac{1}{2})\left\{-\left(\frac{a}{\pi}\right)^{s+\gamma_1+1}\sin\theta\,\sin[(s+1)(\tfrac{1}{2}\pi-\theta)]\right. \\ & \left. +\left(\frac{b}{\pi}\right)^{s+\gamma_1+1}\cos\theta\,\sin[(s+1)\theta\,]\right\}. \end{aligned} \tag{56}$$

The function $U(s,\theta)$ has simple poles at the points $s = -\gamma_1 - 1$ and $s = -k$, $k = 1, 2, \ldots$ due to a simple pole at $z = 1$ of the function $\zeta\,(z, 1/2)$ (with the residue $+1$) and to simple poles at $z = -n$ with $n = 0, 1, 2, \ldots$ of function $\Gamma(z)$ (with the residues $(-1)^n/n!$).

By means of the inverse Mellin transform

$$W_\gamma(\rho,\theta) = \frac{1}{2\pi\mathrm{i}}\int_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} U(s,\theta)\,\rho^{-s}\,\mathrm{d}s \tag{57}$$

and the Cauchy residue theorem, we have for $\rho \to 0$:

$$W_\gamma(\rho,\theta) = \frac{ab}{\pi}\,\Gamma(-\gamma_1)\,\rho^{\gamma_1+1}\,H_{\gamma_1}(\theta) + \sum_{k=1}^{3}\rho^k H_k(\theta) + \mathcal{O}\left(\rho^4\right). \tag{58}$$

Here

$$H_{\gamma_1}(\theta) = \sin\theta\,\sin\left[\gamma_1\left(\theta-\tfrac{1}{2}\pi\right)\right] - \cos\theta\,\sin\gamma_1\theta\,, \tag{59}$$

and

$$H_1(\theta) = \operatorname*{Res}_{s=-1} U(s,\theta) = 0\,,$$

$$H_2(\theta) = \operatorname*{Res}_{s=-2} U(s,\theta) = \zeta\left(\gamma_1,\tfrac{1}{2}\right)\left[b\left(\frac{a}{\pi}\right)^{\gamma_1} + a\left(\frac{b}{\pi}\right)^{\gamma_1}\right]\sin\theta\,\cos\theta\,,$$

$$\begin{aligned} H_3(\theta) = {} & \operatorname*{Res}_{s=-3} U(s,\theta) = -\zeta\left(\gamma_1-1,\tfrac{1}{2}\right)\left[b\left(\frac{a}{\pi}\right)^{\gamma_1-1}\sin^2\theta\,\cos\theta\right. \\ & \left. + a\left(\frac{a}{\pi}\right)^{\gamma_1-1}\sin^2\theta\,\cos\theta\right]. \end{aligned}$$

Remembering the definition (38) of the generalized ζ function, we can write the local expansion of w_γ as

$$
\begin{aligned}
w_\gamma(\rho,\theta) = \ & \mathfrak{Re}\left\{E_\gamma \, \frac{ab}{\pi}\, \Gamma(-\gamma_1)\, \rho^{\gamma_1+1}\, H_{\gamma_1}(\theta)\right\} \\
& +\mathfrak{Re}\left\{E_\gamma \left(b\sum_{m=1}^{\infty}\frac{1}{\alpha_m^{\gamma_1}} + a\sum_{l=1}^{\infty}\frac{1}{\beta_l^{\gamma_1}}\right)\right\} \rho\sin\theta\, \cos\theta \\
& -\mathfrak{Re}\left\{E_\gamma \, b\sum_{m=1}^{\infty}\frac{1}{\alpha_m^{\gamma_1-1}}\right\} \rho^3\, \sin^2\theta\, \cos\theta \\
& -\mathfrak{Re}\left\{E_\gamma \, a\sum_{m=1}^{\infty}\frac{1}{\beta_l^{\gamma_1-1}}\right\} \rho^3\, \sin\theta\, \cos^2\theta + \mathcal{O}\left(\rho^4\right) .
\end{aligned}
\tag{60}
$$

Returning to Equation (42) with asymptotic expressions (51), (53) and (60), by summation of the first line of Equations (26) on m, we finally obtain that whole coefficients of terms with ρ, ρ^2 and ρ^3 in the local expansions of w should *identically* vanish. This conclusion agrees with early established zero values of all up to third partial derivatives of w at the corner point.

Thus, the local representation of w near corner points is

$$
w = \mathfrak{Re}\left\{A_\gamma\, \rho^{\gamma_1+1}\, H_{\gamma_1}(\theta)\right\} + 0(\rho^4)\,, \quad \rho \to 0, \tag{61}
$$

with the amplitude

$$
A_\gamma = \frac{p_0\, ab}{8\pi D}\, E_\gamma\, \Gamma(-\gamma_1)\,. \tag{62}
$$

5. Numerical results and discussion

We chose the plate with $a = 2b$ (this is a typical size of a plate in shipbuilding) in order to compare with some numerical results obtained by Bubnov [27] and Inglis [30] by a similar method. The first two lines in Table 2 give the maximum numerical values of these quantities as well as the total normal reactions. The question mark means that Bubnov [27] doubted these values as well as the negative values of normal reactions near the corner points. Inglis [30] took $M = L = 2$ in his system which is equivalent to Equations (17) and obtained (Figure 2) considerably larger negative values for the edge pressure at the corner points (by about 17 per cent of the maximum value at the clamped side), but he did not comment on these results. It should be noted that the condition (5) of static equilibrium of the plate is satisfied by both numerical data.

In our numerical simulations the representation (16) for the deflection of the plate provides an excellent satisfaction of the boundary conditions (2) even for $M = 4$, $L = 2$ in the infinite system (17). The values of X_m and Y_l from this system can not provide, however, the correct local behaviour of the normal reactions. We have obtained the distribution of the normal and the distribution of the normal reactions similar to those of Inglis (Figure 2), but with the negative values of the pressure at the corner points differing only by about 3 per cent of the maximum values in the middle.

To analize local behaviour we have to turn to the infinite system (26), (27). The infinite sums in the terms with E_γ were calculated numerically with a controlled accuracy. The value of $E_\gamma = -5{\cdot}111 + \mathrm{i}\,1{\cdot}196$ appear to be rather insensitive to increasing the number of equations in the reduced infinite system.

Table 2. Normal reactions at the clamped sides of the rectangular plate, $a = 2b$, according to the method of superposition.

Source	$\frac{V_x(0)}{p_0 b}$	$\frac{V_x(b)}{p_0 b}$	$\frac{V_y(0)}{p_0 b}$	$\frac{V_y(a)}{p_0 b}$	$\frac{R_x}{p_0 ab}$	$\frac{R_y}{p_0 ab}$	$\frac{R_x + R_y}{p_0 ab}$
Bubnov [27]	0·929	0 (?)	1·032	0 (?)	0·502	1·498	2·000
Inglis [30]	0·976	−0·208	1·030	−0·142	0·487	1·513	2·000
System (17)	0·931	−0·035	1·037	−0·036	0·499	1·501	2·000
System (26), (27)	0·928	0·000	1·032	0·000	0·502	1·498	2·000

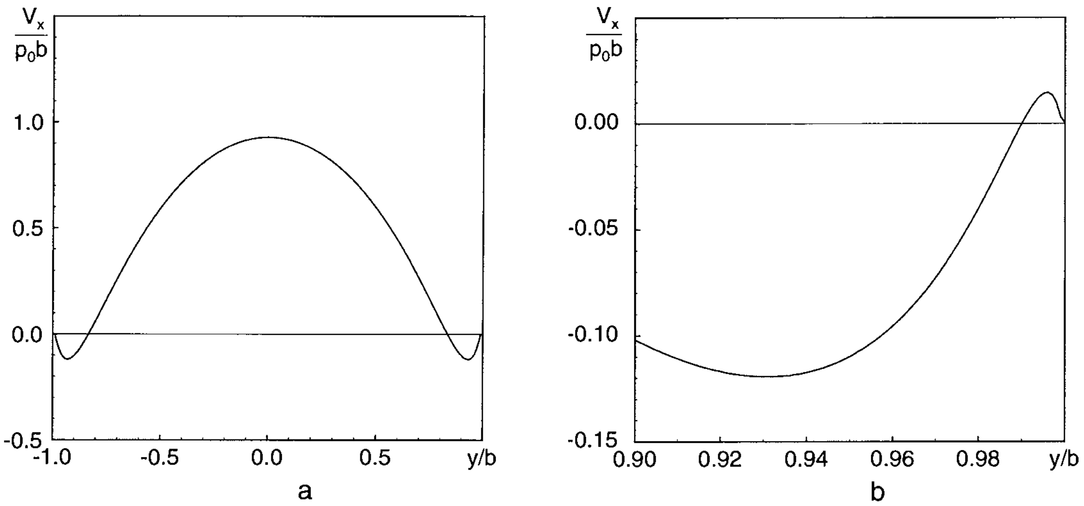

Figure 4. Distribution of the normal reaction V_x along the short clamped side $x = a$ for the plate with aspect ratio $a/b = 2$ according to the present analytical solution. (a) general behaviour; (b) enlarged view near the end of the interval.

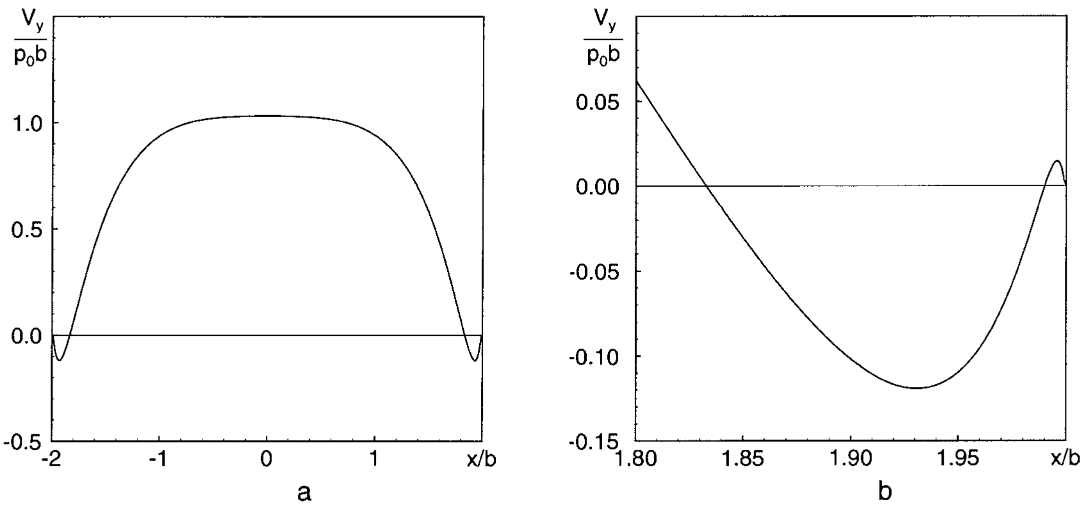

Figure 5. Distribution of the normal reaction V_y along the long clamped side $y = b$ for the plate with aspect ratio $a/b = 2$ according to the present analytical solution. (a) general behaviour. (b) enlarged view near the end of the interval.

Figures 4 and 5 represent the distribution of the normal reactions V_x and V_y along the clamped sides $x = a$ and $y = b$, respectively. In gross view shown in Figures 4(a) and 5(a) these curves resemble curves in Figure 2 with negative values of reactions in the intervals $0{\cdot}83b < |y| < b$ and $0{\cdot}92a < |x| < a$, respectively. But the enlarged pictures presented in Figures 4(b) and 5(b) show a next change of sign of reactions which become positive in the intervals $0{\cdot}986b < |y| < b$ and $0{\cdot}994a < |x| < a$, respectively. Such a behaviour near the ends of intervals completely agrees with the asymptotic expressions (40) and (41) for V_y and V_x. Woinowsky-Krieger [36] presented a similar oscillatory picture of the distribution of normal reactions in a quarter wedge plate bending by a concentrated force. He did not, however, explicitly connect this behaviour with the left-hand side of Equation (21) which appeared in the denominator of an integral representation for normal reactions in the wedge plate. It is worth noting that a similar distribution of contact stresses on an elastic halfplane under a rigid stamp was first obtained by Abramov [54] with another complex root depending upon the elastic properties of the material. The physical attributes of the oscillatory nature of stresses is another matter, and it will remain so as long as the linear elasticity formulation is employed. These aspects do not seem to severely affect the main engineering questions concerning the maximum values of deflection and stresses in a clamped rectangular plate.

Comparing the general picture of the exact distributions of V_x and V_y in Figures 4(a) and 5(a) with the approximate ones presented in Figure 3, we may conclude that all these engineering methods provide a reasonable correspondence in distribution of the normal reaction $V_y(x)$ along the long side $y = b$. The distributions of the normal reaction $V_x(y)$ along the short side $x = a$ reveal less agreement with the exact ones. The Ritz variational approach [23, 44] and its modification done by Galerkin [31] and Biezeno and Koch [48] are in closer correspondence than the various energy approaches [11, 13, 45, 46, 47] which rely, in fact, upon the same idea. This shows the importance of a proper choice of the approximate trial functions when dealing with any of the engineering approaches. Each of these methods is useful when some concrete numerical results regarding deflection and bending moments are needed. These approaches, however, cannot provide reasonable values for the normal pressures and the total normal reactions.

Table 2 shows that Bubnov's [27] data for the maximum values of the normal reactions at the middle of the clamped sides as well as the total reactions completely coincide with ours. Therefore, Bubnov's conclusion about a 10 per cent error in the normal reactions because of poor convergence of the Fourier series near the corner points is too hasty. He simply could not obtain (and, probably, even imagine) the correct asymptotic behaviour of V_x and V_y connected with the *complex* value of γ_1. Inglis's analytical solution [30] can provide the correct negative values of the normal reactions near the end points, but it cannot describe the abovementioned oscillations of the reactions' distributions.

One can argue that any currently available finite-element or boundary-element code could provide an estimate for the edge reactions by specifying the correct order of inter-element contunuity required for the Kirchhoff theory of thin elastic plates with clamped edges. However, even in the recent edition of the fundamental treatise [55] and in the references therein such a possibility has not even been mentioned. The main discussion concerns the accuracy of determination of the maximum deflection in the centre. On the other hand, the boundary-element method [56] even with a refined mesh near the corner points does not provide us with a clear numerical picture of the local behaviour of the normal reactions. We consider our analytical treatment of the *finite* clamped rectangular plate based upon the 'first principles' of the Fourier series could be useful as a benchmark example for testing the accuracy of any numerical scheme.

6. Conclusions

Looking back upon the proposed method of superposition and some results obtained with it, we see that it provides a direct and powerful algorithm for solving the benchmark problem of the bending of a clamped rectangular plate. The algebraic work involved seems rather cumbersome, but the final formulae for local behaviour of the deflection and normal reactions are believed to be rather simple for numerical evaluation and analytical treatment. The advantages of the superposition method – relatively few calculations compared to other approaches and high accuracy of determination of the deflection and normal reactions everywhere including the boundary with corner points, – were elucidated by considering the typical example of the bending of a clamped plate. The algorithm of the improved reduction based upon the asymptotic law for the coefficients allowed us to obtain accurate numerical results using only a few terms in the Fourier series. In the problem in question we see how a mathematical advance in establishing the asymptotic behaviour of the Fourier coefficients permitted us to analyse the engineering characteristics of the clamped rectangular plate, namely, the distribution of normal pressures along the clamped sides. We have confirmed the zero values of normal reactions at the corner points, a suggestion expressed quite a long time ago. We have established a rather complicated oscillatory behaviour of these quantities along the clamped sides near the corner points.

In conclusion, it may be mentioned that the method of superposition given here is equally applicable to the three-dimensional problem of the elastic equilibrium of a finite parallelepiped with mixed boundary conditions at its sides, which is a modification of the famous Lamé problem. But the complete solution of this interesting problem, as well as the challenging question of local behaviour of displacements and shear forces near rims, is still beyond analytical possibilities of modern analysis.

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